	Calculation of Cylinder Radii in the Strip and Pixel Barrel Regions for the Inner Tracker Upgrade		
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Note

Calculation of Cylinder Radii in the Strip and Pixel Barrel Regions for the Inner Tracker Upgrade

This note presents formulae for calculating the ideal radius of a cylinder with a given number of modules and with the required amount of overlap. The aim is to help with parametric layouts, which are easier to change as our understanding of the requirements develops.

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Distribution List:
Everybody

History of Changes

Rev. No.	*Date*	*Pages*	*Made by*	*Description of changes*
A	24/9/07	All	N Hessey	Original document
B	31/10/07	Many	N Hessey	New tables, less approximations.
C	12/2/12	5	N Hessey	Iterate for better r
D	13/07/12	5	N Hessey	Remove wrong r in (13)
E	15/08/12	Many	N Hessey	Add section on inverse problem
F	07/06/17	Many	N Hessey	Add section on N for r

1 Barrel Radii

The radius of a barrel is fixed by the number of sensor rows, the sensor size, the tilt-angle, and the overlap required. It is useful to be able to easily choose the number of rows of sensors needed to fill a cylinder with a desired approximate radius; and to know the actual radius of cylinder needed for that number of rows.

Fig. 1 shows a sensor of width w at a radius r from the beam-line, tilted by an angle ψ . It subtends an angle θ at the beamline. It overlaps with a neighbouring sensor by an angle ϕ. O, A, B, C are vertices of triangles; a and b are lengths of sides of these triangles.

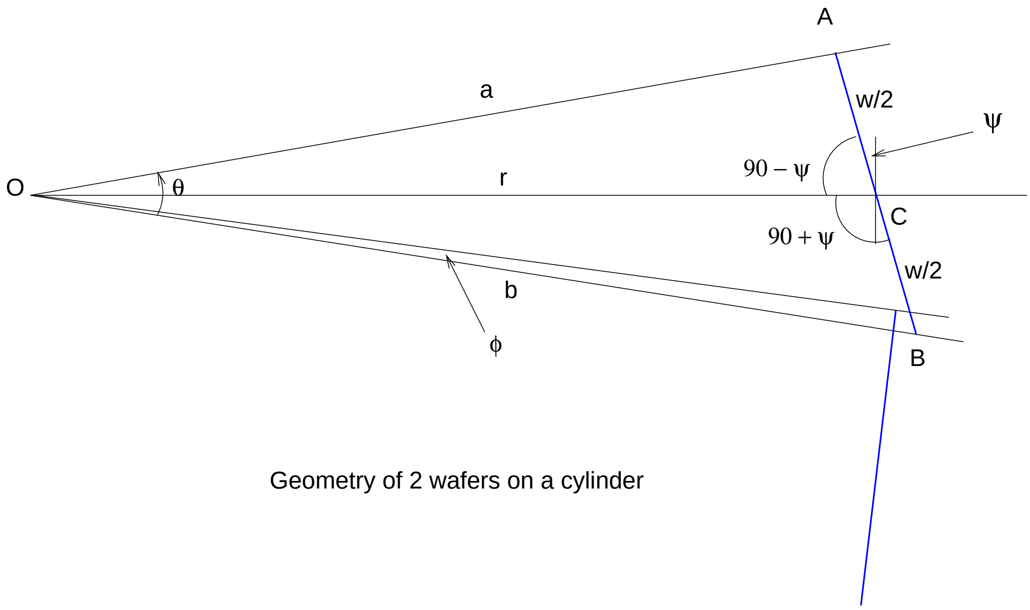

Figure 1: End-view of two sensors on a cylinder, defining variables used.

If sensors overlap each other by a fraction p, then

$$\phi = p\theta \tag{1}$$

and if there are N sensors evenly spaced in a ring, the angle subtended by one sensor is

$$\theta = 2\pi/N(1-p). \tag{2}$$

In fact, p needs to be chosen carefully for each cylinder to ensure hermeticity for tracks down to some minimum p_T (usually set at 1 GeV/c). For now we take p as a known quantity.

Using the cosine rule, triangle OAC gives

$$a^2 = r^2 + (w/2)^2 - 2r(w/2)\cos(90 - \psi), \tag{3}$$

triangle OBC gives

$$b^2 = r^2 + (w/2)^2 - 2r(w/2)\cos(90 + \psi) \tag{4}$$

and triangle OAB gives

$$w^2 = a^2 + b^2 - 2ab\cos\theta. \tag{5}$$

These three equations have three unknowns (a, b, r) and can be solved for r:

$$r^4(1-\cos^2\theta) + r^2(4(w/2)^2\sin^2\psi\cos^2\theta - 2(w/2)^2(1+\cos^2\theta)) + (w/2)^4\sin^2\theta = 0 \tag{6}$$

which is a quadratic equation in r^2 and can be solved in the usual way. (The plus sign in front of the square-root gives the required root).

2 How much overlap?

What should p be? The overlap has two functions: to make layers hermetic for all tracks above some momentum, usually taken as 1 GeV/c; and to allow alignment to be made by having tracks pass through two neighbouring detectors.

For strip-detector alignment, a minimum of 1 % is good: this does not cost much in either money terms or material, and gives enough strips overlapping (about 12) to quickly build up statistics to make alignments frequently. It also avoids putting very stringent requirements on assembly precision: being a couple of tenths of a mm out loses only a small fraction of overlapping strips, and leaves sufficient to do the alignment job.

For pixel alignment, 2 % may be wiser: pixel modules are expected to be about 32 mm wide, so 1 % gives only 320 μm overlap. If you require assembly tolerances not to reduce this by more than 10 %, you have only 16 μm tolerance on module positioning (one module can move one way, its neighbour the other, to reach the limit).

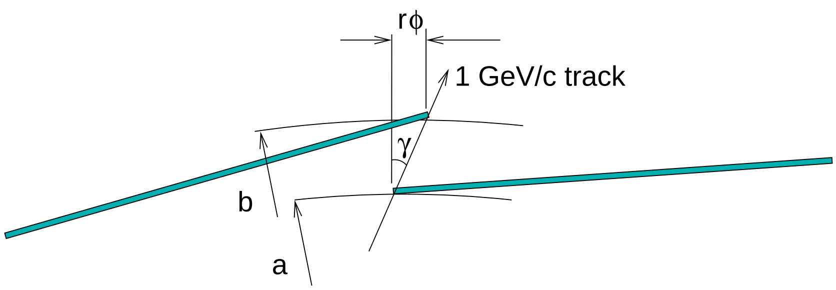

Figure 2: The gap-angle γ related to the overlap $r\phi$ and inner and outer radii of sensors a and b.

The hermeticity requirement needs some calculations. First we find the end-points of the sensors. The end-points are at radii a and b in fig. 2, which can be found from eqs. (3)

and (4) (assuming for now we know r). The circumferential distance between the end-points is approximately $r\phi$, and hence the gap-angle γ between the line joining the two end-points and a radius of the cylinder is given by

$$\tan\gamma = \frac{r\phi}{b-a}. \tag{7}$$

Tracks at a greater angle than this can pass through the gap unrecorded. The angle a track makes when passing through can be estimated assuming a constant magnetic field B, no multiple scattering, and an origin at the centre of the cylinder. The track is a helix, which when projected onto the r, ϕ plane is a circle. Its radius is given by (see e.g. PDG Book)

$$\rho = \frac{p_T}{0.3B}. \tag{8}$$

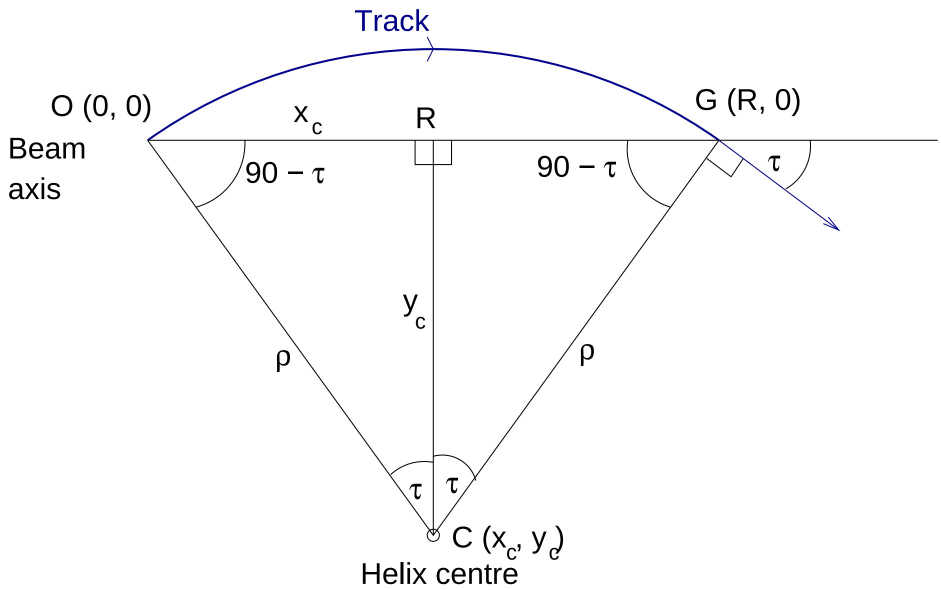

Figure 3: Deriving the angle α that a track makes to the radial direction at a radius R from the origin.

The angle this helix makes to the radial direction at a distance r from the origin can be found considering fig. 3. We choose a coordinate system with the origin O on the beamline and x along the radial line joining the origin to the gap G. The centre C of the track circle is at some point (x_c, y_c) a distance ρ from both the origin and the gap. So triangle OGC is isosceles and $x_c = r/2$. The track has locus $x_c^2 + y_c^2 = \rho^2$ hence $y_c = \sqrt{(\rho^2 - r^2/4)}$. Then the track angle τ at the gap G is given by

$$\tan\tau = x_c/y_c = \frac{r/2}{\sqrt{(\rho^2 - r^2/4)}} = \left[(2\rho/r)^2 - 1\right]^{-1/2}. \tag{9}$$

Setting $\tan\gamma$ and $\tan\tau$ equal gives the boundary of hermeticity. This allows us to determine the required p:

$$\frac{r\phi}{b-a} = \tan\tau \tag{10}$$

From equations (1) and (2)

$$\phi = \frac{2\pi}{N}\frac{p}{1-p} \tag{11}$$

and substituting for ϕ and rearranging gives

$$p = \left[\frac{2\pi r}{N(b-a)\tan\tau} + 1\right]^{-1} \tag{12}$$

and using (9) τ can be eliminated:

$$p = \left[\frac{4\pi\rho}{N(b-a)}\sqrt{1-(r/2\rho)^2} + 1\right]^{-1} \tag{13}$$

which are both exact for this model. However they are cumbersome since r, a, and b depend weakly on p.

Two approximations simplify (12) considerably. Firstly, in (9) the 1 can be neglected: from (8), with $B = 2$ T and $p_T = 1$ GeV/c, $\rho = 5/3$ m, and the worst case r for the cryostat bore is about 1 m so the 1 is less than 10 % of the main term. Hence

$$\tan\tau \approx r/2\rho \tag{14}$$

Secondly $b - a \approx w\sin\psi$: consider going from zero difference when the twist is zero, to one sensor moving out by $0.5w\sin\psi$ and the other moving in by the same amount when the twist becomes non-zero. Alternatively take the equations for a and b and make the approximation $a + b = 2r$.

These two approximations give the fractional overlap for hermeticity from (12)

$$p \approx \left[\frac{4\pi\rho}{Nw\sin\psi} + 1\right]^{-1}. \tag{15}$$

which for a given sensor size w, tilt angle ψ, and required minimum radius of curvature for hermeticity ρ depends only on the number N of sensors in the ring. The overlap needed grows with radius (more sensors). For the pixel region the minimum needed for alignment and assembly tolerence is larger than this hermeticity requirement. For the strip region, the hermiticity requirement is more important.

The approximations above give a slight over-estimate of the maximum radius for a given number of sensors. If desired, this can be improved on with an iterative method. The approximate value of p gives a value for θ from (2). This can be used in (6) to obtain an estimate of r; then estimates of a and b obtained from (3) and (4). Using these a, b, and r in (13) gives a better estimate of the overlap needed for hermeticity, and hence a more accurate estimate of the maximum allowed r from (6) once more. In practice, one iteration is sufficient for a precision of 1 mm; the resulting r is about 7 mm less than the approximate one for $r \approx 1$ m.

3 Results

These formulae (15) and (6) have been incorporated in a spread-sheet to give radii for different conditions. Table 1 gives an example of values of the cylinder radius r and overlap fraction p for different numbers of sensors for strip regions, with 1280 strips of 74.5 μm pitch (ATLAS07 design) and 16° tilt-angle. Table 2 gives results for pixel regions, with 32 mm wide modules and 10° tilt angle.

No discussion has been made of double-sided modules, or tiling with upper and lower modules. In general, the table should be taken as giving the ideal radius of the outer-wafer for double sided modules, and upper modules for a scheme with upper and lower modules. Then all other wafers would be at smaller radii and would have more than sufficient overlap. The radius given is the maximum which will give sufficient overlap for the given number of sensors. In fact, because of the approximations made in the hermiticity calculations, and to allow for assembly tolerances, a design with slightly smaller radii would be sensible; say 0.5 - 1 % less than the maximum calculated when hermiticity is the limiting factor.

Number in ring	Overlap fraction p (%)	Max. radius (mm)
16	2.0	226
18	2.2	255
20	2.4	283
22	2.7	311
24	2.9	338
26	3.2	366
28	3.4	393
30	3.6	421
32	3.9	448
34	4.1	475
36	4.3	501
38	4.6	528
40	4.8	555
42	5.0	581
44	5.2	607
46	5.5	634
48	5.7	660
50	5.9	686
52	6.1	711
54	6.3	737
56	6.6	763
58	6.8	788
60	7.0	813
62	7.2	839
64	7.4	864
66	7.6	889
68	7.9	913
70	8.1	938
72	8.3	963
74	8.5	987
76	8.7	1012
78	8.9	1036
80	9.1	1060

Table 1: Radius to silicon centre and overlap needed for a selection of numbers of sensors, for 95.36 mm active area sensors with 16° tilt angle.

Number in ring N	Overlap fraction p(%)	Radius r (mm)
6	2.0	26.8
8	2.0	37.4
10	2.0	47.6
12	2.0	57.7
14	2.0	67.7
16	2.0	77.7
18	2.0	87.6
20	2.0	97.5
22	2.0	107.4
24	2.0	117.3
26	2.0	127.2
28	2.0	137.1
30	2.0	146.9
32	2.0	156.8
34	2.0	166.7
36	2.0	176.5
38	2.0	186.4
40	2.0	196.2
42	2.0	206.1
44	2.0	215.9
46	2.0	225.8
48	2.0	235.6
50	2.0	245.5
52	2.0	255.3
54	2.0	265.1
56	2.0	275.0
58	2.0	284.8
60	2.0	294.7
62	2.0	304.5
64	2.0	314.3

Table 2: Radius to silicon centre and overlap needed, for a selection of numbers of sensors, for 32 mm active-width sensors with 10° tilt angle (typical for Pixel Upgrade).

4 Calculating the minimum momentum for a given geometry

The inverse problem – "Given the radius, module width and tilt angle, what is the maximum momentum that can sneak through the gap" – is much easier to solve exactly, and serves as a useful check.

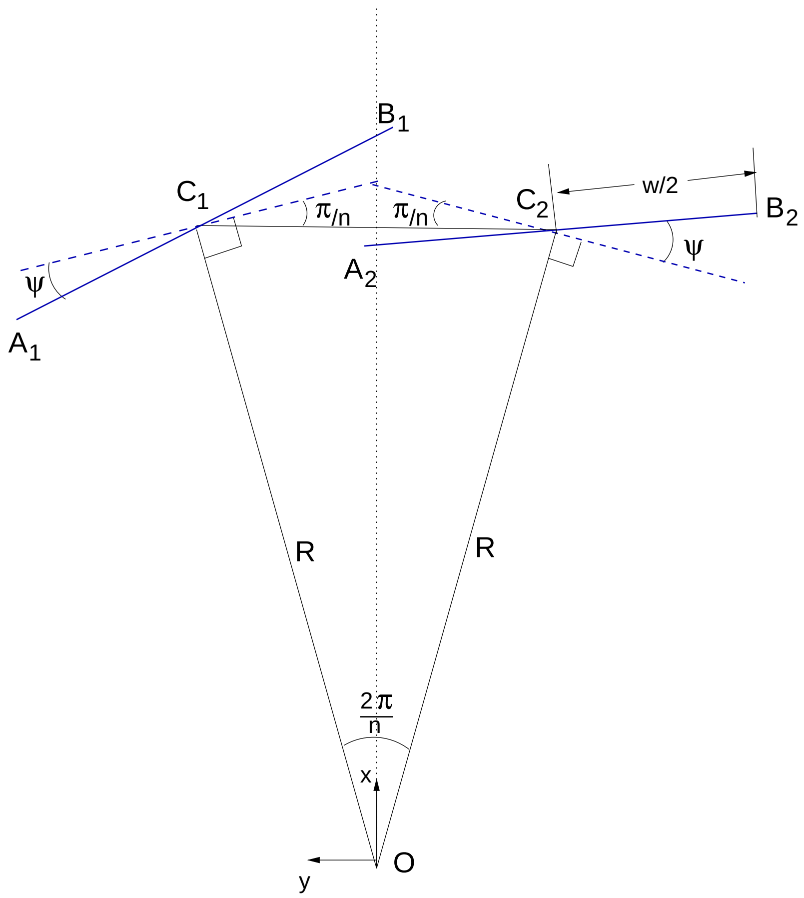

Figure 4: Two neighbouring sensors (blue solid line) showing the geometry used to calculate the minimum hermiticity.

We redraw the situation in fig. 4, showing the symmetry between two neighbouring sensors. The sensors are labelled with subscripts 1 and 2 and have centre points C_i at a radius R and end points A_i (at the inner radius end) and B_i (at the outer radius end). The sensors have a width w and are tilted at an angle ψ to the tangent of a circle of radius R.

The polar coordinates of C_i are clearly $(R, \pm\pi/n)$ so that in Cartesian coordinates

$$C_1 = (R\cos(\pi/n), R\sin(\pi/n)) \tag{16}$$

$$C_2 = (R\cos(\pi/n), -R\sin(\pi/n)) \tag{17}$$

Adding the appropriate displacements of the sensor edges from the centre gives

$$B_1 = (R\cos(\pi/n) + w/2\sin(\psi + \pi/n), R\sin(\pi/n) - w/2\cos(\psi + \pi/n)) \tag{18}$$

$$A_2 = (R\cos(\pi/n) - w/2\sin(\psi - \pi/n), -R\sin(\pi/n) + w/2\cos(\psi - \pi/n)) \tag{19}$$

The radius of the circle passing through the points O, A_2, and B_1 is obtained from the standard formula

$$\rho = \frac{|\boldsymbol{O} - \boldsymbol{B_1}|}{2 \sin \angle O A_2 B_1} \tag{20}$$

$$= \frac{|OB_1|}{2 \sin \angle O A_2 B_1} \tag{21}$$

From the sine-rule

$$\frac{\sin \angle O A_2 B_1}{|OB_1|} = \frac{\sin \angle A_2 O B_1}{|A_2 B_1|} \tag{22}$$

so

$$\rho = \frac{|A_2 B_1|}{2 \sin \angle A_2 O B_1} \tag{23}$$

$$= \frac{\sqrt{(B_{1x} - A_{2x})^2 + (B_{1y} - A_{2y})^2}}{2 \sin(\phi_{A_2} - \phi_{B_1})} \tag{24}$$

This can be simplified, avoiding the use of trigonometric functions, by defining

$$\alpha \equiv \frac{A_{2y}}{A_{2x}} \tag{25}$$

$$\beta \equiv \frac{B_{1y}}{B_{1x}} \tag{26}$$

so that

$$\phi_{A_2} = \arctan \alpha \tag{27}$$

$$\phi_{B_1} = \arctan \beta \tag{28}$$

Straightforward manipulation results in

$$\rho = \sqrt{(B_{1x} - A_{2x})^2 + (B_{1y} - A_{2y})^2} \; \frac{\sqrt{(1+\alpha^2)(1+\beta^2)}}{2(\alpha - \beta)} \tag{29}$$

Finally the transverse momentum of a track which follows a helix with radius ρ is given by equation 8. Note this gives an exact solution to the problem defined in figure 4, without any approximations.

One would like to solve this for R. Squaring both sides, multiplying through by $(A_{2x} B_{1y})^2$, multiplying out, simplifing by working in length units of $w/2$, using trigonometric relations, collecting coefficents of powers of R gives a 6^{th} degree polynomial for R. I could not get the algebra correct. Maybe with correct algebra, one could be lucky and find a way to factorise it. The roots can of course be found numerically, given a set of values for ρ, w, ψ, n, but it seems messier than the method given earlier.

5 How many staves are needed for a given radius?

This problem is much easier than getting r for a given N: given a wafer width w, tilt angle ψ, curvature ρ for tracks at the required momentum limit, what is the number of staves $N(r, \omega, \psi, \rho)$ needed for a given radius r?

Equations 3, 4, and 5 show that given r, ω, and ψ, then a, b, and θ are known. Given r and ρ, equation 9 shows τ is known. So finding $N(r, a, b, \theta, \tau)$ also solves this problem. Starting from 10, substituting for ϕ from 1, then substituting for $p\theta$ from 2 gives

$$\frac{r(\theta - 2\pi)}{b - a} = \tan\tau \tag{30}$$

which can be solved for N:

$$N = \frac{2\pi}{\theta - \frac{b-a}{r}\tan\tau} \tag{31}$$

A calculator for N has been added to the spread-sheet barrelnew.ods.

6 Acknowledgements

I am grateful to Gerard Berbier for providing an engineering drawing used to check (and correct!) the formulae and spreadsheet used.